

FORCING, GENERICITY, AND CBERS

FILIPPO CALDERONI AND DIMA SINAPOVA

ABSTRACT. In this paper we continue the study of equivalence of generics filters started by Smythe in [Smy22]. We fully characterize those forcing posets for which the corresponding equivalence of generics is smooth using the purely topological property of condensation. Next we leverage our characterization to show that there are non-homogeneous forcing for which equivalence of generics is not smooth. Then we prove hyperfiniteness in the case of Prikry forcing and some additional results addressing the problem whether generic equivalence for Cohen forcing is hyperfinite.

1. INTRODUCTION

Let M a countable transitive model of a sufficient fragment of ZFC, and fix a partial order \mathbb{P} in M .

We can define the Polish space $\text{Gen}(M, \mathbb{P})$ of M -generic filters for \mathbb{P} . This space has a basis of clopen sets of the form $N_p = \{G \in \text{Gen}(M, \mathbb{P}) \mid p \in G\}$ for all $p \in \mathbb{P} \cap M$. We say that two generic filters $G, H \in \text{Gen}(M, \mathbb{P})$ are **equivalent** (in symbols, $G \equiv_M^{\mathbb{P}} H$) if and only if they produce the same generic extension. I.e., we define

$$G \equiv_M^{\mathbb{P}} H \quad \iff \quad M[G] = M[H].$$

The equivalence relation $\equiv_M^{\mathbb{P}}$ was first analyzed from the point of view of the Borel complexity theory by Smythe [Smy22]. By the definability of the forcing relations, this is a countable Borel equivalence relation on the Polish space $\text{Gen}(M, \mathbb{P})$. (See [Smy22, Lemma 2.6].) The case where \mathbb{P} is the Cohen forcing \mathbb{C} is particularly compelling, and it is motivated by a long-standing open problem in descriptive set theory. Recall that a Borel equivalence relation is **hyperfinit**e if is the increasing union of a sequence of Borel equivalence relations with finite equivalence classes. While [Smy22, Theorem 3.1] proves that $\equiv_M^{\mathbb{C}}$ is an increasing union of hyperfinite equivalence relations, it is unclear whether $\equiv_M^{\mathbb{C}}$ is hyperfinite. Therefore, the question whether $\equiv_M^{\mathbb{C}}$ is hyperfinite is a specific instance of the (increasing) union problem posed as early as [DJK94]. And people have wondered whether $\equiv_M^{\mathbb{C}}$ constitutes a counterexample. (For a discussion about this, we refer the reader to [Kec24, Page 82]).

In addition to that, Smythe [Smy22, Question 1] asks to determine the Borel complexity of equivalence of generics for other forcings.

In Section 3 we provide a new characterization of smoothness for $\equiv_M^{\mathbb{P}}$. In Sections 4 we prove that $\equiv_M^{\mathbb{P}}$ is hyperfinite in the case of Prikry. In Section 5 we prove some additional results about generic equivalence and we discuss some equivalence relations closely related to $\equiv_M^{\mathbb{C}}$. We believe that these would be useful to address the problem of finding the Borel complexity of $\equiv_M^{\mathbb{C}}$ and other forcing in the future.

2. PRELIMINARIES

2.1. Forcing. Recall that \mathbb{P} is **weakly homogeneous** if for all pairs of conditions q, r , there is an automorphism $e : \mathbb{P} \rightarrow \mathbb{P}$, such that $e(q)$ is compatible with r . Note that this implies that there are generic filters G, H , such that $q \in G, r \in H$ and $V[H] = V[G]$. Also, recall that if, on the other hand G and H are two generic filters for \mathbb{P} with $V[H] = V[G]$, and $\overline{\mathbb{P}}$ is the Boolean completion of the poset, then there is an automorphism $e : \overline{\mathbb{P}} \rightarrow \overline{\mathbb{P}}$, such that $e''G = H$.

Weak homogeneity can be used to detect non-smoothness in the following precise sense. Recall that a poset is **atomless** if and only if every condition has incompatible extensions.

Theorem 2.1 (Smythe [Smy22, Theorem 2.12]). *If \mathbb{P} is atomless and weakly homogeneous, then $\equiv_M^{\mathbb{P}}$ is not smooth.*

In fact, Smythe [Smy22, Lemma 2.11] proved that \mathbb{P} is weakly homogeneous if and only if the action of $\text{Aut}^M(\mathbb{P})$ on $\text{Gen}(M, \mathbb{P})$ is generically ergodic. In Proposition 3.4 below, we show that the sufficient condition in Theorem 2.1 is not necessary.

We will say that \mathbb{P} is **densely weakly homogeneous** if for densely many $p, \mathbb{P}_{\leq p}$ is weakly homogeneous (i.e., the poset $\{q \in \mathbb{P} \mid q \leq p\}$). Clearly, weak homogeneity implies densely weak homogeneity.

For an example of densely weakly homogeneous poset that is not weakly homogeneous, take a lottery sum of non isomorphic weakly homogeneous forcings. Recall that the **lottery sum** of a collection $\{\mathbb{P}_i : i \in I\}$ of forcing posets is the poset $\bigoplus_{i \in I} P_i := \{\langle i, p \rangle : i \in I \text{ and } p \in \mathbb{P}_i\} \cup \{\mathbb{1}\}$, ordered with $\mathbb{1}$ weaker than everything else and $\langle i, p \rangle \leq \langle j, q \rangle$ when $i = j$ and $p \leq_{\mathbb{P}_i} q$. Then for all $\langle i, p \rangle$, the forcing below $\langle i, p \rangle$, the poset $\mathbb{P} \downarrow \langle i, p \rangle$ is isomorphic to $\mathbb{P}_i \downarrow p$, which is just \mathbb{P}_i by homogeneity, and therefore weakly homogeneous, and so the full forcing is densely weakly homogeneous. On the other hand, since the \mathbb{P}_i 's are not isomorphic, the full forcing is not weakly homogeneous.

2.2. Countable Borel equivalence relations. An equivalence relation E is countable if every E -class is countable. It is finite if every E -class is finite.

Let $X = 2^{\mathbb{N}}$. Then the **eventual equality relation**

$$x E_0 y \iff \exists m \forall n \geq m (x_n = y_n).$$

It is immediate that E_0 is hyperfinite because $E_0 = \bigcup i \in \mathbb{N} F_i$, where $x F_i y \iff x_n = y_n$ for all $n \geq i$. Clearly each F_i class has cardinality 2^i .

Let Γ be a countable group with the discrete topology. If X is a Polish space and $a: \Gamma \times X \rightarrow X$ is a continuous action of Γ on X we say that X is a **Polish Γ -space**. In which case, we denote by $\mathcal{R}(\Gamma \curvearrowright X)$ the corresponding orbit equivalence relation. Given a Polish Γ -space X , a point $x \in X$ is **condensed** if x is an accumulation point of its orbit. That is, for all open neighborhoods U of x , $|U \cap [x]| \geq 2$. In the sequel, we make use of the following well-known fact. For a proof see e.g. [CC24, Proposition 2.2].

Fact 2.2. *Let X be a Polish Γ -space. The orbit equivalence relation $\mathcal{R}(\Gamma \curvearrowright X)$ is not smooth if and only if there is a condensed point.*

3. CHARACTERIZATION OF SMOOTHNESS

The main goal in this section is to present a characterization of non-smoothness for $\equiv_M^{\mathbb{P}}$. (Theorem 3.3 below.) First we spell out condensation in the case when X is $\text{Gen}(M, \mathbb{P})$ and $\Gamma = \text{Aut}^M(\mathbb{P})$. Recall that $N_p = \{G \in \text{Gen}(M, \mathbb{P}) \mid p \in G\}$. It is crucial for us that the set $\{N_p \mid p \in \mathbb{P} \cap M\}$ is a basis for the space $\text{Gen}(M, \mathbb{P})$. (For the details see [Smy22, Lemma 2.3].) Therefore we can refer to the basic (cl)open sets N_p , rather than referring to all open subset of the space.

Definition 3.1. An M -generic filter $G \in \text{Gen}(M, \mathbb{P})$ is **condensed** if for all N_p with $G \in N_p$, there is $H \neq G$, such that $M[H] = M[G]$ and $H \in N_p$.

Definition 3.2. Let \mathbb{P} be a forcing poset in M . For any $p \in \mathbb{P}$ define the statement (\dagger_p) as follows:

(\dagger_p) For all $p' \leq p$, there are incompatible $q, r \leq p'$,
such that there exists distinct generic filters G, H ,
such that $q \in G, r \in H$ and $V[H] = V[G]$.

In other words condition (\dagger_p) says that we can find instances of homogeneity densely often below p . Clearly, densely weakly homogeneous implies (\dagger_p) for all p . We will show that this is strictly weaker than being densely weakly homogeneous.

Theorem 3.3. *For any forcing poset \mathbb{P} , we have $\equiv_M^{\mathbb{P}}$ is not smooth if and only if (\dagger_p) holds for some $p \in \mathbb{P}$.*

Proof. For the forward direction, if $\equiv_M^{\mathbb{P}}$ is not smooth, there is a condensed filter $G \in \text{Gen}(M, \mathbb{P})$. It follows that (\dagger_p) holds for all $p \in G$. (Use Definition 3.1.)

For the right to left direction, assume (\dagger_p) for some $p \in \mathbb{P}$. Let G be a generic filter with $p \in G$. We want to show that G is condensed. Namely, for all $p' \in G$ there is $H \neq G$ with $M[G] = M[H]$ and $H \ni p'$.

Fix $p' \in G$. By strengthening if necessary, we may assume $p' \leq p$. First we prove the following density lemma.

Claim. The set $\mathcal{D} = \{q : \exists e \in \text{Aut}(\mathbb{P}) \cap M \text{ such that } e(q) \perp q, e(q) \leq p'\}$ is dense in \mathbb{P} below p' .

Proof. Given $q \leq p' \leq p$ by (\dagger_p) we can find incompatible conditions $q', r \leq q$ such that

$$q' \in H_1 \quad r \in H_2 \quad M[H_1] = M[H_2].$$

Then there is $e \in \text{Aut}(\bar{\mathbb{P}}) \cap M$ such that $e''H_1 = H_2$. So, $e(q')$ and r are compatible. Then let $r^* \leq e(q'), r$ and denote by $q^* = e^{-1}(r^*)$. It follows that $q^* \leq q'$ and also that q^* and $e(q^*) = r^*$ are incompatible. Then $q^* \in \mathcal{D}$. \square

Choose $q \in \mathcal{D} \cap G$ so that $q \leq p'$. Let $e \in \text{Aut}(\mathbb{P}) \cap M$ witness that $q \in \mathcal{D}$. By letting $H = e''G$ we get $M[G] = M[H]$ and since q and $e(q)$ are incompatible it follows that G, H are different. And since both q and $e(q)$ are below p' , both G, H contain p' , as desired. \square

Using condensation we show the following:

Proposition 3.4. *There is a poset \mathbb{P} that is not weakly homogeneous, but $\equiv_M^{\mathbb{P}}$ is not smooth.*

Towards the proof of Proposition 3.4, we will define \mathbb{P}_ℓ such that \mathbb{P}_ℓ is not densely weakly homogeneous, but it satisfies (\dagger_p) for all $p \in \mathbb{P}_\ell$. The idea is to take lottery sums of nonisomorphic weakly homogeneous forcings in a tree like fashion. Let \mathbb{P}_0 be a weakly homogeneous forcing poset. For ease of exposition we take a constant sequence of forcing $\{\mathbb{P}_0^i : i \in \mathbb{N}\}$ with $\mathbb{P}_0^i = \mathbb{P}_0$. Let $\{\mathbb{P}_1^i : i \in \mathbb{N}\}$ be a sequence of nonweakly homogeneous forcings such that no \mathbb{P}_1^i projects¹ into \mathbb{P}_1^j for $i \neq j$. For nonempty $\sigma \in 2^{<\omega}$ let \mathbb{P}_σ be $\prod_{n < |\sigma|} \mathbb{P}_{\sigma(n)}^n$, so that $\mathbb{P}_{01} = \mathbb{P}_0 \times \mathbb{P}_1^1, \mathbb{P}_{11} = \mathbb{P}_1^0 \times \mathbb{P}_1^1, \mathbb{P}_{110} = \mathbb{P}_1^0 \times \mathbb{P}_1^1 \times \mathbb{P}_0$, etc. Finally we define \mathbb{P}_ℓ as follows: Conditions in \mathbb{P}_ℓ are of the form (σ, p) for some $\sigma \in 2^{<\omega}$ and $p \in \mathbb{P}_\sigma$, and

$$(\sigma, p) \leq (\tau, q) \quad \iff \quad \sigma \supseteq \tau \text{ and } p \upharpoonright \tau \leq_{\mathbb{P}_\tau} q.$$

Lemma 3.5. *\mathbb{P}_ℓ is not densely weakly homogeneous, but (\dagger_p) holds for all p . Therefore, $\equiv_{M, \mathbb{P}_\ell}$ is not smooth.*

Proof. Since \mathbb{P}_1^i are not weakly homogeneous, we can find conditions $q_1, r_1 \in \mathbb{P}_1^i$ such that for all \mathbb{P}_1^i -generics $G_1 \ni q_1$ and $H_1 \ni r_1$ we have $V[G_1] \neq V[H_1]$. It follows that the poset \mathbb{P}_ℓ is not weakly homogeneous because any condition (σ, p) can be extended to incompatible conditions $(\sigma \hat{\ } 1, p \hat{\ } q_1)$ and $(\sigma \hat{\ } 1, p \hat{\ } r_1)$. Then if G, H are \mathbb{P}_ℓ -generic containing $(\sigma \hat{\ } 1, p \hat{\ } q_1)$ and $(\sigma \hat{\ } 1, p \hat{\ } r_1)$ respectively, and

¹Here \mathbb{P}_0 projects onto \mathbb{P}_1 means that whenever G is generic for \mathbb{P}_0 , one can define a generic for \mathbb{P}_1 in $V[G]$.

$V[G] = V[H]$, then the projections of G, H on the $|\sigma|$ coordinate must produce the same model.

On the other hand, any condition (σ, p) can be extended to conditions $(\sigma \frown 0, q), (\sigma \frown 0, r)$ such that $q, r \in \mathbb{P}_{\sigma \frown 0}$ and $q \upharpoonright |\sigma| = r \upharpoonright |\sigma| = p$. Using homogeneity of \mathbb{P}_0 we can find generics containing these conditions that produce the same extension. Therefore, we conclude that $\mathcal{R}(\text{Aut}(\mathbb{P}_\ell) \curvearrowright \text{Gen}(\mathbb{P}_\ell, M))$ is not smooth by Lemma 3.3, and neither $\equiv_{M, \mathbb{P}_\ell}$ is. \square

4. PRIKRY IS HYPERFINITE

Let κ be a measurable cardinal and fix a normal measure U on κ (i.e. U is a κ -complete non principal normal ultrafilter on κ). In this section \mathbb{P} will be the a Prikry forcing at κ with respect to U . I.e. conditions are of the form (s, A) , where s is a finite increasing sequence of points in κ and $A \in U$. We will refer to set in U as “measure one”. For $(s, A) \in \mathbb{P}$, we call s the stem of the condition, and $|s|$ is the length. We say q is a direct extension of p , denoted $q \leq^* p$ if $q \leq p$ and they have the same stem.

Let M be a countable model satisfying enough of ZFC, containing κ and \mathbb{P} .

A generic filter for \mathbb{P} induces a generic Prikry sequence $\vec{\alpha} = (\alpha_n : n < \omega)$ cofinal in κ , and it is actually defined from it as follows: given such a sequence $\vec{\alpha}$, define G as the set of conditions (s, A) , such that for some n , $s = (\alpha_0, \dots, \alpha_{n-1})$ and for all $i \geq n$, $\alpha_i \in A$. We say that $\vec{\alpha}$ is a **generic sequence** if the filter induced from it is generic. We will be identifying generic filters with the corresponding generic Prikry sequences.

By identifying M with its transitive collapse, we can assume that M is transitive without any loss of generality.

Theorem 4.1. $\equiv_M^{\mathbb{P}}$ is hyperfinite.

We will use the following fact about Prikry forcing.

- Fact 4.2.** (1) If $\vec{\alpha}$ and $\vec{\beta}$ are both \mathbb{P} -generic sequences over M , then $M[\vec{\alpha}] = M[\vec{\beta}]$ if and only if $\vec{\alpha}$ and $\vec{\beta}$ coincide on a tail.
- (2) (The Prikry lemma). For any condition p and a dense open set D , there is a direct extension $q \leq^* p$ and $k < \omega$, such that every $r \leq q$ of length k is in D .
- (3) Prikry forcing is weakly homogeneous. In particular, for any \mathbb{P} -generic G over M and any condition $p \in M$, there is a \mathbb{P} -generic H over M , such that $p \in H$ and $M[G] = M[H]$.

Fact 4.2(1) implicitly follows from the results [GKK20]. Since this work is unpublished, for completeness we provide a proof below.

Proof. One direction is clear. For the other direction, suppose for contradiction that $\vec{\alpha}$ and $\vec{\gamma}$ are two generic Prikry sequences over M , such that $M[\vec{\alpha}] = M[\vec{\gamma}]$,

but the symmetric difference of these sequences is infinite. It follows that there is a subsequence of $\vec{\gamma}$, which is disjoint from $\vec{\alpha}$. Let $\vec{\beta}$ enumerate this subsequence. Note that $\vec{\beta} \in M[\vec{\alpha}]$.

Let p in the generic filter induced by $\vec{\alpha}$, force that $\vec{\beta}$ is disjoint from the generic Prikry sequence.

For each n , let D_n be the dense set of conditions deciding a value for $\dot{\beta}_n$. By the Prikry lemma, there is a direct extension $p' \leq^* p$, and $k < \omega$, such that every k -step extension of p' is in D_n . I.e., for every $r \leq p'$ with length $\text{lh}(r) = \text{lh}(p) + k$, there is $\delta < \kappa$, such that $r \Vdash \delta = \dot{\beta}_n$.

Claim 4.2.1. *Fix n , and suppose that k is minimal such that the above holds for some $p' \leq^* p$ and D_n , and denote $p' = (h, A)$. Then there is a measure one set $B \subset A$, such that for all increasing s in B^k , if δ_s is the unique such that $(h \frown s, B \setminus \max(s) + 1) \Vdash \delta = \dot{\beta}_n$, we have that $\max(s) < \delta_s$.*

Proof. Suppose otherwise. Since p' forces that $\dot{\beta}_n$ is not on the Prikry sequence, each $\delta_s \notin s$. Let $\phi: [A]^k \rightarrow \{0, 1\}$ be the function defined as follows

$$\phi(s) = \begin{cases} 0 & \text{if } \max(s) < \delta_s \\ 1 & \text{if } \max(s) > \delta_s. \end{cases}$$

By measurability, ϕ is homogeneous on a measure one set, namely, there is $B \subset A$, $B \in U$, such that for all k , $\phi \upharpoonright [B]^k$ is constant. So, for all k -length stems s taken from B , $\delta_s < \max(s)$. Then for each $t \in [B]^{k-1}$, the function $g_t: \alpha \mapsto \delta_{t \frown \alpha}$ is regressive. By Fodor's Theorem (e.g., see [Jec03, Theorem 8.7]), there is a measure one set $B_t \subset B$ and $\delta_t < \kappa$, such that for all s of the form $t \frown \alpha$ with $\alpha \in B_t$, $\delta_s = \delta_t$. Let $B' = \Delta B_t = \{\alpha \mid \alpha \in \bigcap_{\max(t) < \alpha} B_t\}$. Then if $r \leq (h, B')$ is any condition with length $\text{lh}(p) + k - 1$; denote $r = (h \frown t, C)$. We may assume that $\max(t) < \min(C)$. Then $C \subset B_t$, and so $r \Vdash \dot{\beta}_n = \delta_t$. Contradiction with the minimality of k . \square

Using the above claim and inductively applying the Prikry lemma² for each n , we can find a condition $p' = (h, A) \leq^* p$, and a sequence $(k_n : n < \omega)$, such that:

- (1) for each n , every k_n -step extension of p' decides the value of $\dot{\beta}_n$,
- (2) for each stem s of points in A of length k_n , setting δ_s to be such that $(h \frown s, A \setminus \max(s) + 1) \Vdash \dot{\beta}_n = \delta_s$, we have that $\delta_s > \max(s)$.

Now, let G be the generic filter induced by $\vec{\alpha}$. Since we can do this densely often, we can find such a condition in G . Now go back to $M[G]$. Let n , be such that $\beta_n \in A$. Let $q \leq p$ be the (unique) weakest k_n -step extension of p in G . Denoting $q = (h \frown s, A \setminus \max(s) + 1)$, we have that $q \Vdash \dot{\beta}_n = \delta_s$. Since $q \in G$, then $\delta_s = \beta_n \in A$. And since $\max(s) < \delta_s$, we can further extend q to $r = (h \frown s \cup \{\delta\}, A \setminus \delta + 1)$. But then r forces that $\dot{\beta}_n$ is on the Prikry sequence. Contradiction with our initial assumption. \square

²cf. Fact 4.2(2)

Proof of Theorem 4.1. Since M is countable, we identify $(\kappa \cap M)^\omega$ with the Baire space. Roughly, speaking we encode $\equiv_M^{\mathbb{G}}$ into a hyperfinite equivalence relation on the standard Borel space of \mathcal{G} of generic Prikry sequences in $(\kappa \cap M)^\omega$. Precisely, since there is a one-to-one Borel function $f: \text{Gen}(M, \mathbb{P}) \rightarrow (\kappa \cap M)^\omega$ associating to any filter the corresponding generic Prikry sequence, let $\mathcal{G} = f''\text{Gen}(M, \mathbb{P})$ which is injective image of a Borel set, therefore Borel.

Working in V , enumerate $M \cap \kappa$ by $(\gamma_n : n < \omega)$. Define E_n as follows: for two M -generic Prikry sequences \vec{x} and \vec{y} in \mathcal{G} , let $\vec{x} E_n \vec{y}$ if and only if

- $x_k = y_k$, for all $k \geq n$, and
- $\vec{x} = \vec{y}$ or for all $k < n$, $x_k, y_k \in \{\gamma_i \mid i \leq n\}$.

Then each E_n is a finite equivalence relation. In fact, $|\vec{x} E_n| \leq n!$. Moreover, $E_n \subset E_{n+1}$, and $\equiv_M^{\mathbb{P}} = \bigcup_n E_n$. So $\equiv_M^{\mathbb{P}}$ is hyperfinite, and in particular $\equiv_M^{\mathbb{P}} \sim_B E_0$. \square

5. TOWARDS HYPERFINITENESS AND MUTUAL GENERICITY

Let M be the ground model, and let $\mathbb{P} \in M$ be a poset. Let $X = \text{Gen}(M, \mathbb{P})$, the set of M -generics for \mathbb{P} . Note that $X \notin M$, but $X \subset 2^{\mathbb{P}}$ and $2^{\mathbb{P}} \in M$. Throughout this section we analyze the equivalence relation $\equiv_M^{\mathbb{P}}$ on X and discuss hyperfiniteness. We simplify our notation by letting E be $\equiv_M^{\mathbb{P}}$.

Following the analysis in Smythe, we have that E is generated by the countable group $\text{Aut}^M(\overline{\mathbb{P}})$, where $\overline{\mathbb{P}}$ is the Boolean algebra completion of \mathbb{P} . (E.g., see [Smy22, Theorem 2.16].) Let $\{\gamma_n \mid n < \omega\}$ enumerate the automorphisms in $\text{Aut}^M(\overline{\mathbb{P}})$. For each n , let Γ_n be the group generated by $\{\gamma_i \mid i \leq n\}$. Let E_{Γ_n} be the induced orbit equivalence relation on $2^{\mathbb{P}}$. (We note that the full $2^{\overline{\mathbb{P}}}$ may not be a Polish space, so we take the restriction just to \mathbb{P} .) Note that while the full enumeration of the automorphisms is not in M , for each n , both Γ_n and E_{Γ_n} are in M . Moreover $E_{\Gamma_n} \subseteq E_{\Gamma_{n+1}}$ and $E = \bigcup_n E_{\Gamma_n}$.

Let $\{I_n \mid n < \omega\} \in M$ be an increasing family of infinite subsets of ω , such that $\bigcup_n I_n = \omega$ and $I_{n+1} \setminus I_n$ is infinite. Find Borel involutions $\{g_k \mid k < \omega\}$ on $2^{\mathbb{P}}$ (possibly with repetitions), such that for each n , $\{g_k \mid k \in I_n\} \in M$ and $E_{\Gamma_n} = \bigcup_{k \in I_n} \text{graph} g_k$. We do this by applying Feldman-Moore for E_{Γ_n} in M for each n . Then, also $E = \bigcup_k \text{graph} g_k$.

Next we define finite Borel equivalence relations E_s , for $s \in \omega^{<\omega}$, so that if $s \subseteq s'$, then $E_s \subseteq E_{s'}$ and each $E_s \subset E$ with a cascade argument as in [KM04, Theorem 12.1]. More precisely, we first fix a Borel linear ordering $<$ on $2^{\mathbb{P}}$. Then, for each $n < \omega$, let $f_k(x) = \min_{<} \{x, g_k(x)\}$. And for $s = \langle s_0, \dots, s_{l-1} \rangle \in \omega^{<\omega}$, set

$$x E_s z \iff f_{s_{l-1}} \circ \dots \circ f_{s_1} \circ f_{s_0}(x) = f_{s_{l-1}} \circ \dots \circ f_{s_1} \circ f_{s_0}(z).$$

Note that each $E_s \in M$ and the sequence $\langle E_s \mid s \subset I_n \rangle \in M$ for each $n \in \omega$.

Next, fix n and let $z \in E_{\Gamma_n}$ and $s \in I_n^{<\omega}$. Let

$$x' = f_{s_{l-1}} \circ \dots \circ f_{s_1} \circ f_{s_0}(x) \quad \text{and} \quad z' = f_{s_{l-1}} \circ \dots \circ f_{s_1} \circ f_{s_0}(z)$$

where $l = |s|$. Then $x' E_{\Gamma_n} z'$, and so $g_k(x') = z'$, for some $k \in I_n$. Let $s' = s \frown k$. Then we have that $z E_{s'} x$. Take say the minimum such s' and denote it by $s'_{x,z}$. Note that the function $(x, z, s) \mapsto s'_{x,z}$ is in M .

Now, for any x, z such that $x E z$, and $s \in \omega^{<\omega}$, let $s'_{x,z} = s'_{x,z}$, where n is the least such that $z E_{\Gamma_n} x$ and $s \subset I_n$. Of course, the map $\Phi: (x, z, s) \mapsto s'_{x,z}$ is not in M , however, Φ restricted to $s \in I_n^{<\omega}$ and x, z with $z E_{\Gamma_n} x$ is in M .

For x, z , such that $x E z$, let $D_{x,z} = \{s'_{x,z} \mid s \in \omega^{<\omega}\}$. This is a dense subset of the poset $(\omega^{<\omega}, \supseteq)$ (which can be identified with $\text{Add}(\omega, 1)$). And define:

$$C_{x,z} = \bigcup_{s \in \omega^{<\omega}} \mathcal{N}_{s'_{x,z}}.$$

Here we use the notation $\mathcal{N}_s := \{y \in \omega^\omega \mid s \sqsubset y\}$. Clearly, $C_{x,z}$ is dense open. For each x , let $C_x = \bigcap_{z, w \in [x]_E} C_{w,z}$. Then C_x is comeager in ω^ω . Also, for each n and x, z , such that $x E_{\Gamma_n} z$, define

$$C_{x,z}^n = \bigcup_{s \in I_n^{<\omega}} \mathcal{N}_{s'_{x,z}}.$$

Furthermore, for each x , let $C_x^n = \bigcap_{z, w \in [x]_{E_{\Gamma_n}}} C_{w,z}^n$. Note that C_x^n is comeager in $I_n^{<\omega}$.

We will say that $y \in \omega^\omega$ is **N -generic** for some model N , if it is generic for the poset $(\omega^{<\omega}, \supseteq)$ over N . Recall that $X = \text{Gen}(M, \mathbb{P})$ is the set of M -generics for some poset \mathbb{P} .

Definition 5.1. We say that $f: X \rightarrow \omega^\omega$ is an **invariant E -mutual generic function** for \mathbb{P} if and only if

- (i) $f(x)$ is $M[x]$ -generic for all $x \in X$; and
- (ii) $x E_M^{\mathbb{P}} y \implies f(x) = f(y)$.

Theorem 5.2. *If there is an invariant \mathbb{P} -mutual generic function $g: X \rightarrow \omega^\omega$, then $\equiv_M^{\mathbb{P}}$ is hyperhyperfinite.*

Proof. Let E denote $\equiv_M^{\mathbb{P}}$ and suppose that g is such a function.

For each n and $y \in \omega^\omega$, let E_y^n be defined as above, with respect to the Borel involutions generating E_{Γ_n} . Now, define E_n by setting

$$x E_n z \iff x E z \text{ and } x E_{g(x)}^n z.$$

Then E_n is a hyperfinite relation because $E_n = \bigcup_k F_{n,k}$ where $x F_{n,k} z \iff x E z$ and $x E_{g(x)}^n \upharpoonright k z$.

Moreover, if $x E_{\Gamma_n} z$, then since $g(x)$ is $M[x] = M[z]$ -generic and $D_{x,z}^n \in M[x]$, we have that $x E_{g(x)}^n z$. It follows that $E_M^{\mathbb{P}} = \bigcup_n E_n$, and for each n , $E_n \subseteq E_{n+1}$. \square

Corollary 5.3. *There is no invariant \mathbb{B} -mutual generic function, when \mathbb{B} is the random forcing.*

Proof. The equivalence relation $\equiv_M^{\mathbb{P}}$ is not Fréchet amenable by [Smy22, Theorem 4.4]. Since every hyperfinite equivalence relation is amenable, every hyperhyperfinite equivalence relation is Fréchet amenable (e.g., see Proposition 9.11(v) of [Kec24]). Therefore the corollary is an immediate consequence of Theorem 5.2. \square

Proposition 5.4. *Suppose that $y \in \omega^\omega$ and $x \in X$ are M -mutually generic. (I.e. $y \times x$ is M -generic for $\omega^{<\omega} \times \mathbb{P}$.) Suppose also that $\{g_n \mid n < \omega\} \in M[y]$ are Borel involutions, such that $\equiv_M^{\mathbb{P}} = \bigcup_n \text{graphs}(g_n)$. Then $[x]_{\equiv_M^{\mathbb{P}}} = [x]_{E_y}$. Namely, $\equiv_M^{\mathbb{P}}$ restricted to $\text{Gen}(\mathbb{P}, M[y])$ is hyperfinite.*

Proof. As above, denote $\equiv_M^{\mathbb{P}}$ by E . The inclusion $[x]_E \supseteq [x]_{E_y}$ is clear. For the nontrivial direction, suppose $z \in X$ is such that $z E x$. Then y is $M[z]$ generic (as $M[x] = M[z]$, $D_{x,z} \in M[z]$, and so for some $s \in D_{x,z}$, $s \sqsubset y$). It follows that $z \in [x]_{E_s} \subseteq [x]_{E_y}$. The hyperfiniteness of E restricted to $\text{Gen}(\mathbb{P}, M[y])$ follows since every $\text{Gen}(\mathbb{P}, M[y])$ is mutually generic with y . \square

Next we look at the Cohen forcing $\mathbb{C} = \text{Add}(\omega, 1)$ to add one real. Recall that this poset is isomorphic to $(\omega^{<\omega}, \supseteq)$. Let $h: 2^{<\omega} \rightarrow \omega^{<\omega}$ code that isomorphism, $h \in M$. Note that then for any $M' \supseteq M$, $x \in 2^\omega$ is M' -generic iff $h''x$ is M' -generic. In slight abuse of notation we use $h(x)$ to denote $h''x$.

Let evens, odds denote the sets of even and odd natural numbers respectively. For $x \in 2^\omega$, let $x_l = x \upharpoonright \text{evens}$ and $x_r = x \upharpoonright \text{odds}$. Note that if x is M -generic, then both x_l and x_r are also M -generic and moreover, they are mutually generic, i.e., x_l is $M[x_r]$ -generic and vice versa. Finally, Let $h: 2^{\text{evens}} \rightarrow \omega^\omega$ be some isomorphism in M . Note that if $x \in 2^{\text{evens}}$ is M -generic, and $y = h(x)$, then so is y .

Let E denote $\equiv_M^{\mathbb{C}}$. By Feldman-Moore, E is induced by a countable group generated by some Borel involutions $\{g_n \mid n < \omega\}$. Let $N \supset M$ be a countable model, such that the $\{g_n \mid n < \omega\} \in N$.

Define the following subequivalence relations of E (both defined on X):

- $x E^* z$ iff $x_l = z_l$ and $x_r E z_r$.
- $x E^N z$ iff $x = z$ or $x E z$, $x_l = z_l$, and x is N -generic.

Note that if there is an invariant \mathbb{P} -mutual generic function, then $E \leq_B E^*$.

Lemma 5.5. *E^N is hyperfinite.*

Proof. We first make some observations. First note that for every $x \in X$, x_l and x_r are mutually generic. Suppose now that $x E^N z$, $x \neq z$, and denote $w = x_l = z_l$. Then $M[x] = M[w][x_r] = M[w][z_r] = M[z]$. So $x_r \equiv_{M[w]}^{\mathbb{P}} z_r$. By assumption x and therefore z are also N -generic. It follows that x_r, z_r are N -generic, and w is $N[x_r] = N[z_r]$ -generic.

Define a sequence of equivalence relations E_n , $n < \omega$ as follows. Set

$$x E_n z \iff x E^N z \text{ and } x_r E_{y \upharpoonright n} z_r,$$

for $y = h(x_l)$. Then the E_n 's are increasing finite equivalence relations. To show that their union is E^N , suppose $x E^N z$. Let $w = x_l = z_l$ and $y = h(w)$. Then the dense set D_{x_r, z_r} is in $N[x_r]$ and by mutual genericity, y is $N[x_r]$ -generic. It follows that $y \upharpoonright n \in D_{x_r, z_r}$ for some n , and so $x_r E_{y \upharpoonright n} z_r$. \square

We end this section with some questions, that may be stepping stones for settling whether equivalence of generics is hyperfinite for the Cohen forcing and other posets.

Question 5.6. *For which posets \mathbb{P} , is there an invariant \mathbb{P} -mutual generic function? Is there such a function for the Cohen poset?*

Note that by Corollary 5.3 there is no invariant mutually generic function for the Random forcing. On the other hand, if $\equiv_M^{\mathbb{P}}$ is finite, then one can define such a function.

Question 5.7. *Suppose that $\equiv_M^{\mathbb{P}}$ is hyperfinite. Must there be a \mathbb{P} -invariant mutual generic function?*

REFERENCES

- [CC24] Filippo Calderoni and Adam Clay. Condensation and left-orderable groups. *Proc. Amer. Math. Soc. Ser. B*, 11:579 – 588, 2024.
- [DJK94] Randall Dougherty, Stephen Jackson, and Alexander S. Kechris. The structure of hyperfinite borel equivalence relations. *Transactions of the American Mathematical Society*, 341(1):193–225, 1994.
- [GKK20] Moti Gitik, Vladimir Kanovei, and Peter Koepke. Intermediate models of prikry extensions. Preprint, available at https://www.researchgate.net/publication/228378916_Intermediate_models_of_Prikry_generic_extensions, 2020.
- [Jec03] Thomas Jech. *Set theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.
- [Kec24] Alexander S. Kechris. *The Theory of Countable Borel Equivalence Relations*. Cambridge Tracts in Mathematics. Cambridge University Press, 2024.
- [KM04] Alexander S. Kechris and Benjamin Miller. *Topics in orbit equivalence relations*. Lecture notes in Mathematics. Springer-Verlag Berlin Heidelberg, 2004.
- [Smy22] Iian B. Smythe. Equivalence of generics. *Arch. Math. Log.*, 61(5–6):795–812, jul 2022.

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, HILL CENTER FOR THE MATHEMATICAL SCIENCES, 110 FRELINGHUYSEN RD., PISCATAWAY, NJ 08854-8019
Email address: filippo.calderoni@rutgers.edu

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, HILL CENTER FOR THE MATHEMATICAL SCIENCES, 110 FRELINGHUYSEN RD., PISCATAWAY, NJ 08854-8019
Email address: dima.sinapova@rutgers.edu